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Uniqueness and minimum theorems for a multifield model of brittle solids

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Abstract

Some minimum theorems potentially useful to construct numerical schemes related to quasi-static evolution of damage in brittle elastic solids are proposed. The approach is that of multifield theories, with a second-order damage tensor describing the microcrack density. The use of damage entropy flux and damage pseudo-potential are both investigated.

Keywords: Continuum damage mechanics; Microcracks; Second-order damage tensor; Multifield theories; Damage entropy flux; Pseudo-potential

1. Introduction

In material science and engineering the analysis of damage evolution has become important in the last decades. It is well known that damage is mainly due to microscopic defects which cause the progressive deterioration of elastic properties of the material, a deterioration which is the product of irreversible changes of the material texture at the *microlevel*.

In *brittle* and *quasi-brittle* solids the damage growth, which results in the degradation of the material elastic properties as well as in a overall strength reduction, is essentially characterized by the nucleation, coalescence or growth of microcracks and microvoids while in *ductile* solids damage is driven by the evolution of microshear bands and dislocations.

In continuum damage mechanics internal variables are usually used to take into account distributed defects on the microlevel, in order to obtain the expression of the degraded material constitutive law,

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for instance the weakened elastic moduli (see Kachanov, 1986; Krajcinovic, 1996; Lemaitre, 1996). New internal variables in the coupled damage-elastoplasticity theory, related to microscopic changes in the *representative material element*, have been defined for example by Souchet (2003). It is worth noting that an efficient physical interpretation of a damage variable rests upon the precise identification of microstructural mechanisms "hidden" behind the macroscopic response.

Classical damage models, however, present shortcomings when applied to numerical simulations, especially when working with materials with softening. Basically they are sensitive to the finite element mesh adopted, a circumstance which can be avoided by introducing non-local terms: accordingly the evolution of damage variables depends on the current value of state variables in the point and in a zone surrounding this point. These non-local damage theories "have been recognized as a theoretically clean and computationally efficient approach" (see Borino et al., 2003 and the references quoted therein).

Regularizations techniques need in general the definition of enriched continuum models, in which long-range and short-range interactions between material elements are considered, and not neglected at all as in classical continuum models. In this sense a new approach, based on a multifield description of damage and able to eliminate the mesh sensitivity, has been recently developed (see Frémond and Nedjar, 1996; Markov, 1995; Mariano, 1995, 1999; Mariano and Augusti, 1998; Stumpf and Hackl, 2003). The basic idea consists in the introduction of new independent kinematic variables describing the *microcracked* state in each material element at each point **X**. The material point together with the information relative to its material texture (the microcrack patterns in a damage model) constitute the representative material element.

This model of microcracked bodies results in a weakly non-local damage model, as the constitutive dependence of mechanical interactions on the gradient of damage variable is considered. As a consequence strain localization phenomena appear even if constitutive relations are linear and microcracks are in the elastic phase without growth (numerical results concerning the localization of deformation are contained in Mariano and Stazi (2001) and Mariano et al. (2002)).

In this paper we rephrased the model presented in Mariano and Augusti (1998) for brittle solids, such as ceramics, concrete and rocks, and discuss it within the setting of infinitesimal strains. We proposed some theorems concerning uniqueness and minimum properties of the solution of the damage evolution problem for linear elastic microcracked solids. Two distinct models for the irreversible damage growth in brittle elastic solids are considered: the one based on the concept of damage entropy flux (Sections 2–4) and the other based on the definition of a damage pseudo-potential (Section 5), obtaining different evaluations of the dissipation function.

2. Multifield description of damage distribution and evolution in brittle solids

We recall some basic features and equations of multifield description of damage in microcracked bodies. For the general treatment of this way to study damage evolution, consisting in the nucleation of new microcracks and the propagation and clustering of existing ones, we refer to the model of Mariano and Augusti (1998), for *microcracked hard matter*, and that of Mariano (1999), for *microcracked soft matter*, in which the contribution of the microcracks to the overall deformation is prominent.

In this paper only hard brittle materials undergoing infinitesimal deformations are considered. Current and reference configurations of the body $\mathscr{B}$ are 'almost' coincident with each other and we use the symbol B to indicate the place occupied by the body and assume that it is a fit region of Euclidean three-dimensional space $\mathscr{E}$. Moreover we denote by $\mathbf{x}$ the placement in B of a material element X of reference position $\mathbf{X}$ and by t the instant time.[1]

[1] In what follows $\mathscr{V}$ denotes the space of translations associated to $\mathscr{E}$, Lin the space of linear transformations of $\mathscr{V}$ in $\mathscr{V}$, Sym the subspace of Lin of symmetric second-order tensors, Skw its orthogonal complement and Psym the set of all positive-definite elements of Sym. The inner product between two elements $\mathscr{T}$ and $\mathscr{P}$ of a linear space is defined as $\mathscr{T} \cdot \mathscr{P} = \mathscr{T}_{k\ldots m}\mathscr{P}_{k\ldots m}$.

The complete configuration of a microcracked body is given by the following mapping k:

$$\mathbf{X} \mapsto (\mathbf{x}(\mathbf{X}), \Xi(\mathbf{X})) \tag{2.1}$$

which associates to each material patch at $\mathbf{X}$ its actual position $\mathbf{x}$ and a second-order tensor Ξ which accounts for its material texture, i.e., collects the information about the microcracked state characterizing the material point. The variable Ξ is assumed as an *order parameter*, i.e., an independent kinematic descriptor useful to geometrically characterize the damage within a multifield model.

We remark that the mapping $k_{\mathbf{x}}: \mathbf{X} \mapsto \mathbf{x}(\mathbf{X})$ is the standard deformation of the body, i.e., a continuous and piecewise continuously differentiable bijection, preserving the orientation. The spatial field $\Xi(k_{\mathbf{x}}^{-1}(\mathbf{x}))$ is the value of the order parameter at $\mathbf{x}$. We assume the mapping $\Xi(\cdot)$ continuous and piecewise continuously differentiable too.

A complete motion is a time-parameterized family of mappings $k^t: (\mathbf{X}, t) \mapsto (\mathbf{x}(\mathbf{X}, t), \Xi(\mathbf{X}, t))$, with $k_{\mathbf{x}}^t : (\mathbf{X}, t) \mapsto \mathbf{x}(\mathbf{X}, t)$ the standard motion.

When working with the second-order tensor approximation of the crack density distribution (see Lubarda and Krajcinovic, 1993) the density of cracks embedded in the plane with normal $\mathbf{n}$ through a material point $\mathbf{X}$ is given in its second-order approximation by the following function:

$$m(\mathbf{X}, \mathbf{n}, t) = \Xi(\mathbf{X}, t) \cdot (\mathbf{n} \otimes \mathbf{n}) \tag{2.2}$$

with the *second-order crack density tensor* defined as follows:

$$\Xi = \frac{15}{8\pi}\mathbf{D} - \frac{3}{2}\bar{m}\mathbf{I}. \tag{2.3}$$

In (2.3)

$$\bar{m} = \frac{1}{\Omega}\int_{\Omega} m(\mathbf{n})\mathrm{d}\Omega, \quad \mathbf{D} = \int_{\Omega} m(\mathbf{n})(\mathbf{n} \otimes \mathbf{n})\mathrm{d}\Omega \tag{2.4}$$

are respectively the average crack density and the second-order damage tensor. In (2.4) $\Omega = 4\pi$, the entire solid angle.[2] Of course the symmetry of $\mathbf{D}$ implies that of the order parameter. If $\mathscr{M}$ denotes the manifold of substructural configurations, i.e. the differentiable manifold (without boundary) where the order parameter takes values, for the microcrack density Ξ it results $\mathscr{M} \equiv \mathrm{Sym}$.[3]

We recall the basic equations of the elastic problem of microcracked continua in small deformations regime and linear behavior in the elastic phase: the strain–displacement relation

$$\mathbf{E} = \mathrm{sym}\,\nabla\mathbf{u} \quad \text{in } B, \tag{2.5}$$

the balance equations (of momentum, micromomentum and related boundary conditions concerning generalized tractions)

$$\mathrm{div}\,\mathbf{T} + \mathbf{b} = 0 \quad \text{in } B, \tag{2.6}$$

$$\mathrm{div}\,\mathsf{S} - \mathbf{Z} = 0 \quad \text{in } B, \tag{2.7}$$

$$\mathbf{Tn} = \mathbf{t}, \quad \mathsf{Sn} = \mathsf{t} \qquad \text{on } \partial B, \tag{2.8}$$

[2] In the case of two-dimensional problems $\Omega = 2\pi$.

[3] We observe that the cracks density m could also takes negative values. In fact when the second-order approximation of microcracks distribution, as the fourth-order as well, is used to approximate some typical three and two-dimensional crack patterns, the emergence of regions characterized by negative crack density (*anticrack regions*) should be expected, as a consequence of the approximation process "of discontinuous, narrow band width distributions of cracks by continuous distributions provided by tensors" (Lubarda and Krajcinovic, 1993).

and the balance of moment of momentum, which results in the following prescription on the skew part of the stress tensor:

$$\mathbf{e}[\mathbf{T}] = \mathsf{A}^{\mathrm{T}}[\mathbf{Z}] + (\nabla \mathsf{A}^{\mathrm{T}})[\mathsf{S}] \quad \text{in } B. \tag{2.9}$$

In previous equations $\nabla \equiv \partial_{\mathbf{X}}$ is the gradient operator,[4] $\mathbf{u} = \mathbf{x} - \mathbf{X}$ the displacement vector, $\mathbf{E}$ the infinitesimal deformation tensor, $\mathbf{T}$ the standard (Cauchy) stress, $\mathbf{t}$ the traction vector, $\mathbf{b}$ the body force, S a third-order tensor mechanically representing the *microstress*, t a second-order tensor representing the *generalized traction* (that is the surface traction exerted on a microcrack of a material element by those close to it), $\mathbf{Z}$ a second-order tensor representing the *internal self-force* (that is the action between microcracks in the same material element). Moreover in (2.9) $\mathbf{e}$ is Ricci's tensor and A is the linear operator of the action of the proper orthogonal group Orth^+ on $\mathscr{M}$ (for the general theory of continua with substructure see the works of Capriz (1985, 1989, 2000) and Mariano (2001); see also Capriz and Virga (1990)). We here assume null body forces acting on microcracks by the external world, i.e. the microcracks exchange interactions with the external world only through the surface-like boundary of the body. Furthermore the conservation of mass is implicitly satisfied.

Remark 1. In writing (2.5) we do not take into account the direct influence of microcracks on the deformation of the body as we concern with hard matter. In soft matter, on the contrary, the overall linearized deformation is given by

$$\mathbf{E}_t = \operatorname{sym} \nabla \mathbf{u} + \mathbf{E}_{\Xi}$$

with the second addendum representing the contribution of microcracks. This decomposition is analogous to that typical of infinitesimal plasticity or to the definition of the relative strain in micromorphic continua.

If the following dependence on the state variables for the Helmoltz free energy ψ is assumed:

$$\psi = \widehat{\psi}(\nabla \mathbf{u}, \Xi, \nabla \Xi, \vartheta) \tag{2.10}$$

with ϑ the absolute temperature, and if the measures of interaction $\mathbf{T}$, S and $\mathbf{Z}$ depend also on the same variables as ψ, then from the Clausius–Duhem inequality the following constitutive relations are obtained:

$$\mathbf{T} = \frac{\partial \psi}{\partial \nabla \mathbf{u}}, \quad \mathsf{S} = \frac{\partial \psi}{\partial \nabla \Xi}, \quad \mathbf{Z} = \frac{\partial \psi}{\partial \Xi}, \tag{2.11a}$$

$$\eta = -\frac{\partial \psi}{\partial \vartheta}, \tag{2.11b}$$

the last one concerning with the entropy density η.

Remark 2. When relations (2.11a) are used into (2.9), this accordingly results in the condition for the free energy to be *frame-indifferent*. Thus (2.9) is a constitutive prescription for the skew part of the Cauchy stress, the symmetric one being

$$\operatorname{sym} \mathbf{T} = \partial_{\nabla \mathbf{u}} \psi - \frac{1}{2} \mathbf{e}(\mathsf{A}^{\mathrm{T}}[\mathbf{Z}] + (\nabla \mathsf{A}^{\mathrm{T}})[\mathsf{S}]). \tag{2.12}$$

In the sequel we consider only the *isothermal* case for which ϑ is a parameter and we restrict our analysis to a *linearized* setting.

If the measures of substructural interactions S and $\mathbf{Z}$ are linear functions, by substituting them into (2.9), since $A = -\mathbf{e}\Xi - \Xi({}^t\mathbf{e})$ (cf. Mariano and Augusti, 1998, Eq. (12)), the skew-symmetric part of the macrostress $\mathbf{T}$ would be non-linear in the products $\Xi \nabla \mathbf{u}$, $\Xi\Xi$, $\Xi \nabla \Xi$, $\nabla \Xi \nabla \mathbf{u}$, $\nabla \Xi \nabla \Xi$. Since we deal only with

[4] Which can be confused with the actual gradient operator in the case of small deformations. Besides, the time derivative and the gradient operator commute. Throughout the paper ∂_y means partial derivative with respect to the argument y.

the linearized setting, we forget this skew part corresponding to an higher order contribution, so that we chose the following quadratic form of the internal energy density $w = \psi + \eta\vartheta$:

$$w = \hat{w}(\nabla \mathbf{u}, \Xi, \nabla \Xi) = \frac{1}{2}(\mathsf{C}[\nabla \mathbf{u}] \cdot \nabla \mathbf{u} + \mathsf{S}[\nabla \Xi] \cdot \nabla \Xi + \mathsf{K}[\Xi] \cdot \Xi + 2\mathsf{H}[\Xi] \cdot \nabla \mathbf{u} + 2\mathsf{h}[\nabla \Xi] \cdot \nabla \mathbf{u} + 2\mathsf{f}[\nabla \Xi] \cdot \Xi). \tag{2.13}$$

In (2.13) appropriate constitutive tensors have been defined, reflecting the particular material symmetry characterizing the body in the reference placement.[5] By neglecting the skew part of $\mathbf{T}$ or, alternatively, assuming that constitutive tensors C, H and h could have symmetry properties such that the right-hand side of (2.9) vanishes, it results that $\hat{w}(\nabla \mathbf{u}, \Xi, \nabla \Xi) = \hat{w}(\mathbf{E}, \Xi, \nabla \Xi)$ and consequently

$$\begin{aligned} \mathbf{T} &= \widehat{\mathbf{T}}(\mathbf{E}, \Xi, \nabla \Xi) = \partial_{\mathbf{E}} w = \mathsf{C}[\mathbf{E}] + \mathsf{H}[\Xi] + \mathsf{h}[\nabla \Xi], \\ \mathbf{Z} &= \widehat{\mathbf{Z}}(\mathbf{E}, \Xi, \nabla \Xi) = \partial_{\Xi} w = \mathsf{K}[\Xi] + \mathsf{H}^{\mathrm{T}}[\mathbf{E}] + \mathsf{f}[\nabla \Xi], \\ \mathsf{S} &= \widehat{\mathsf{S}}(\mathbf{E}, \Xi, \nabla \Xi) = \partial_{\nabla \Xi} w = \mathsf{S}[\nabla \Xi] + \mathsf{h}^{\mathrm{T}}[\mathbf{E}] + \mathsf{f}^{\mathrm{T}}[\Xi], \end{aligned} \tag{2.14}$$

according to which the elastic energy density (2.13) is an homogeneous quadratic form in $\mathbf{E}$, Ξ and $\nabla \Xi$, assumed positive-definite (we have indicated with a superscript T the transposition; for a fifth-order tensor h, for example, it results: $\mathsf{h}^{\mathrm{T}}[\mathbf{E}] \cdot \nabla \Xi = \mathsf{h}[\nabla \Xi] \cdot \mathbf{E}$).

If the coupling tensors H and h are both zero, the damage does not effect the macrostress in the real body. On the contrary, $(2.14)_1$ geometrically exhibits the influence of damage on the effective stress $\widehat{T}(\mathbf{E}, \Xi, \nabla \Xi)$.

Remark 3. If we consider, for instance, the local model by disregarding the influence of the damage gradient $\nabla \Xi$, we get an internal variable model in reality; in fact we obtain $\mathsf{S} = 0$ and from (2.7), as a consequence, $\mathbf{Z} = 0$. By combining then $(2.14)_{1,2}$ we have the following *damaged elasticity law* expressing the stress tensor in terms of the deformation for a microcracked material:

$$\mathbf{T} = \mathsf{C}^{(\Xi)}[\mathbf{E}], \tag{2.14b}$$

in which the tensor of *damaged elastic coefficients*, K being invertible,

$$\mathsf{C}^{(\Xi)} := \mathsf{C} - \mathsf{H}\mathsf{K}^{-1}\mathsf{H}^{\mathrm{T}} \tag{2.14c}$$

is such that the following *weakening condition* is valid:

$$\|\mathsf{C}^{(\Xi)}[\mathbf{E}]\| \leqslant \|\mathsf{C}[\mathbf{E}]\| \quad \forall \mathbf{E} \in \mathrm{Sym}. \tag{2.14d}$$

Here and in the sequel we write $\|\mathscr{T}\|$ for the norm of a tensor $\mathscr{T}$ of any order in its space. More precisely, let $\mathscr{X}$ and $\mathscr{Y}$ be two given Banach spaces and $\mathscr{L}(\mathscr{X}, \mathscr{Y})$ denote the space of all bounded operators from $\mathscr{X}$ to $\mathscr{Y}$. Let

$$\|\mathscr{T}\| = \sup_{x \in \mathscr{X}, \|x\|=1} \|\mathscr{T}(x)\|$$

be the norm in this space.[6]

[5] In particular C, H and K are fourth-order tensors while f, h and finally S are respectively fifth-order and sixth-order tensors. The introduction of new constitutive tensors related to the material substructure, behind the elasticity tensor C of classical elasticity, allows the macroscopic constitutive description of the influence of microcracks on the behavior of the body. The brittle deformation caused by nucleation and growth of microcracks, provided it results in an anisotropic damage distribution, influences the anisotropy of the material response. Next relations (2.14) mathematically describe the physical anisotropy due to the fact that the evolution of microcracks, and defects in general, depends on the direction of the applied stress.

[6] Of course $\|\cdot\|$ is an algebra norm, $\mathscr{T}(x) = \mathscr{T}x$ when linear operators are considered and for second-order tensors ($\mathscr{X} \equiv \mathrm{Lin}$) this definition produces the Hilbertian norm $\|\mathbf{A}\| = (\mathbf{A} \cdot \mathbf{A})^{1/2}$.

Equation (2.14b) expresses the *strain equivalence principle* (see, e.g., Lemaitre, 1996), i.e.

$$\mathbf{E} = \mathsf{C}^{(\Xi)^{-1}}[\mathbf{T}] = \mathsf{C}^{-1}[\widetilde{\mathbf{T}}], \tag{2.14e}$$

$\widetilde{\mathbf{T}} = \mathsf{C}\mathsf{C}^{(\Xi)^{-1}}[\mathbf{T}]$ being the *effective stress tensor*. A similar expression can be achieved also for the global model expressed by (2.14), as all the equations are linear and thus for a *microcracked elastic material* the microcrack density tensor always obeys the functional dependence

$$\Xi = \mathsf{F}(\mathbf{E}) \tag{2.14f}$$

with F a linear functional.

It must be noted from (2.13), moreover, that damage generally modifies the elastic strain energy, which is a realistic feature.

It is possible to account for irreversible damage evolution associated with changes in microcracks configuration by defining a function $\mathbf{h}^{\mathrm{D}}$, called the *damage entropy flux*, which "accounts for the dissipation connected to the irreversible growth of microcracks" (Mariano and Augusti, 1998). The divergence of this function represents the configurational entropy due to the irreversible rearrangement of microcracks:

$$f = \operatorname{div}\mathbf{h}^{\mathrm{D}} \geqslant 0. \tag{2.15}$$

Basic motivations about the possibility of the introduction of natural entropies in the state space of damaged bodies have been investigated by Mariano (1997).[7] According to local criteria of damage growth of Mariano and Augusti (1998, §5), we say that there is nucleation, coalescence or growth of microcracks in a neighborhood of a point $\mathbf{x}$ if the two inequalities hold true

$$\dot{\Xi} \cdot (\mathbf{n} \otimes \mathbf{n}) \geqslant 0, \quad q = -\dot{\mathbf{h}}^{\mathrm{D}} \cdot \mathbf{n} \leqslant 0 \tag{2.16}$$

for a direction $\mathbf{n}$ emanating from $\mathbf{X}$, the first inequality representing the growth of microcrack density in a neighborhood of the point $\mathbf{X}$ while the second the growth of the dissipation along the direction $\mathbf{n}$ of microcracks expansion (which obviously results in a growth of microcrack density along $\mathbf{n}$). Moreover from $(2.16)_1$ it results that the time-rate of change of the microcrack density $\dot{\Xi}$ is a positive-semidefinite tensor, i.e. or $\dot{\Xi} = 0$ or $\dot{\Xi} \in \mathrm{Psym}$.[8]

The dissipation (density) function $D(\mathbf{X}, t)$ associated with the irreversible growth of microcracks is given by

$$D = \vartheta \operatorname{div}\mathbf{h}^{\mathrm{D}} \geqslant 0. \tag{2.17}$$

If the following constitutive assumption is made:

$$\mathbf{h}^{\mathrm{D}} = \hat{\mathbf{h}}^{\mathrm{D}}(\mathbf{E}, \Xi, \nabla\Xi), \tag{2.18}$$

then an evolution equation for the microcracked state concerning the rate $\dot{m} = \dot{\Xi} \cdot (\mathbf{n} \otimes \mathbf{n})$ can be obtained by using the *principle of maximum dissipation*. We focus our attention on microcrack density growth and in particular on quasi-static processes related to the microcracked state evolution, characterized by $\|\ddot{\mathbf{u}}\| \approx 0,\ \|\ddot{\Xi}\| \approx 0$, so that inertial effects are totally negligible.

[7] In general it is possible to assume the following additively decomposition $\mathbf{h} = \mathbf{h}(\text{heat flux}) + \mathbf{h}(\text{substructural fields})$, which can be easily justified if linear constitutive relations are assumed. In isothermal process, like those we are interested in, the entropy flux depends exclusively on damage.

[8] In brittle damaging materials in the sense of Marigo (2000) the condition that the damage parameter should be a not decreasing function of time during damage evolution is one of the request for the Drucker–Ilyushin stability postulate be valid.

By means of the maximum dissipation principle we maximize f under the condition $(2.16)_2$. This constrained optimization problem is satisfied by the condition

$$\frac{\partial L}{\partial \Xi} = 0, \quad \text{with } L = -\mathrm{div}\,\mathbf{h}^{\mathrm{D}} + \dot{\lambda} q, \tag{2.19}$$

together with the following Kuhn–Tucker condition[9]

$$\dot{\lambda} \geqslant 0, \quad \dot{\lambda} q = 0, \tag{2.20}$$

$\dot{\lambda}$ being the Lagrange multiplier. From $(2.19)_1$ we get

$$\frac{\partial(\mathrm{div}\,\mathbf{h}^{\mathrm{D}})}{\partial \Xi} = \dot{\lambda} \frac{\partial q}{\partial \Xi} \tag{2.21}$$

which, granted (2.18), results in the following non-linear evolution equation for $\dot{\Xi}$:

$$\partial^2_{\Xi\Xi}(\mathbf{h}^{\mathrm{D}} \cdot \mathbf{n})[\dot{\Xi}] + \partial^2_{\Xi\nabla\Xi}(\mathbf{h}^{\mathrm{D}} \cdot \mathbf{n})[\nabla\dot{\Xi}] = -\dot{\lambda}^{-1}\partial_{\Xi}(\mathrm{div}\,\mathbf{h}^{\mathrm{D}}) - \partial^2_{\Xi\mathbf{E}}(\mathbf{h}^{\mathrm{D}} \cdot \mathbf{n})[\dot{\mathbf{E}}], \tag{2.22}$$

the expression of $\dot{\lambda} > 0$ being obtainable by means of the consistency condition (or condition of persistency)

$$\dot{\lambda}\dot{q} = 0. \tag{2.23}$$

The inequality (2.15) represents a failure criteria. When the equality sign is valid, namely $f = 0$, then the state transformation is reversible and since damage is irreversible (cf. $(2.16)_1$), it means that microcracks are in elastic phase and there is neither nucleation nor growth.[10] Condition (2.15) is equivalent to the monotonicity condition of the yield function in classical plasticity. By using (2.18) into (2.15), the equation $f(\mathbf{E}, \Xi, \nabla\Xi) = 0$ represents in the state space of a microcracked body, with generic element $s \equiv (\mathbf{E}, \Xi, \nabla\Xi)$, a surface which deforms when damage progresses. Of course, the following *condition* applies *between variations of f and directions along which the microcrack density changes*:

$$\partial_{\nabla\Xi} f(\mathbf{E}, \Xi, \nabla\Xi) \cdot \nabla\widetilde{\Xi} \geqslant 0 \quad \forall \nabla\widetilde{\Xi}, \tag{2.24}$$

and *the same derivative of f is upper limited.*

In the next two Sections we will prove the uniqueness of the solution of the evolution problem just stated, indicated in the following by P_1, and an important variational characterization of the isothermal linear elastic response of a body with irreversible growth of microcracks.

We observe that when the one-dimensional counterpart of this model is considered, the second-order microcrack density tensor Ξ reduces to a scalar-valued function, the microvoid density or porosity, and the results obtainable agree to those known in classical damage mechanics literature for isotropic forms of damage (cf. the models of Markov (1995) and Frémond and Nedjar (1996) whose numerical computations show no mesh sensitivity).

We also observe that some interesting results were recently obtained in damage mechanics, by using another approach different from the multifield one but with some features in common. In particular Del Piero and Truskinovsky (2001), in the framework of one-dimensional elasticity with non-convex energy, make use of a non-concave surface energy to model the formation of the so called *process zone*, characterized by an infinite number of infinitesimal cracks. This diffuse zone of non-differentiability can be appropriately described by using the theory of *structured deformations* of Del Piero and Owen (1993, 2000), which is a powerful tool to describe a body characterized by minute geometrical discontinuities.

[9] The Kuhn–Tucker optimality condition (2.20) is sometimes called the complementary slackness condition (see, e.g., Smith, 1974).
[10] Continuing with remarks of footnote 5, we observe that, fixed a point $\mathbf{X}$, the transition $f = 0 \rightarrow f > 0$ during the evolution of damage can be interpreted as a rupture of symmetry.

3. Uniqueness

We here consider the global uniqueness of the solution to the evolution problem P_1, considered in the previous Section, relative to nucleation, coalescence or growth of microcracks in a linear elastic body.

Our uniqueness theorem and its proof follow the methodology of the one given in DeSimone et al. (2001), and developed within the classical scheme of internal variables, for the one-dimensional damage evolution problem of materials characterized by softening behavior. The context is obviously different, also because their model is an internal damage variable one: it does not account for the gradient of Ξ, the balance (2.7) and the damage entropy flux $\mathbf{h}^{\mathrm{D}}$.

Let us consider the following hypotheses. Let $\dot{\mathbf{E}}$, $\dot{\Xi}$ and $\nabla\dot{\Xi}$ be bounded

$$\|\dot{\mathbf{E}}(\mathbf{X},t)\| < +\infty, \quad \|\dot{\Xi}(\mathbf{X},t)\| < +\infty, \quad \|\nabla\dot{\Xi}(\mathbf{X},t)\| < +\infty \tag{3.1a}$$

with

$$\|\dot{\Xi}(\mathbf{X},t)\| \leqslant c_1\|\Xi(\mathbf{X},t)\|, \tag{3.1b}$$

c_1 being a positive real constant. Moreover, let us assume the following boundness condition on the damage entropy flux:

$$0 < c \leqslant \|\partial_{\Xi} f(\mathbf{E},\Xi,\nabla\Xi)\| \leqslant C < +\infty \tag{3.2}$$

with c and C two constants. According to this condition the function $f(\mathbf{E},\cdot,\nabla\Xi)$ grows at least linearly with Ξ. Let finally suppose that there exist another positive constant $\bar{c}$ such that

$$\|\partial^2_{\mathbf{E}\Xi} f(\mathbf{E},\Xi,\nabla\Xi)\| \leqslant \bar{c}, \quad \|\partial^2_{\Xi\Xi} f(\mathbf{E},\Xi,\nabla\Xi)\| \leqslant \bar{c}, \quad \|\partial^2_{\nabla\Xi\Xi} f(\mathbf{E},\Xi,\nabla\Xi)\| \leqslant \bar{c}. \tag{3.3}$$

We now state the result:

Theorem 1. *If (3.1)–(3.3) apply then the evolution problem P_1 has a unique solution.*

Proof. The first step is to prove the uniqueness of the solution $s \equiv (\mathbf{E}, \Xi, \nabla\Xi)$ in the substructural field Ξ and its gradient. To this purpose, let us fix $\mathbf{E}$ and suppose that two distinct solutions $(\mathbf{E}, \Xi_1, \nabla\Xi_1) \neq (\mathbf{E}, \Xi_2, \nabla\Xi_2)$ of the evolution problem, satisfying the same initial conditions at time $t = 0$, correspond to the same external data $\mathscr{F} \equiv (\mathbf{b}, \mathbf{t}|_{\partial B}, \mathrm{t}|_{\partial B})$.

Granted the positiveness of f and condition (3.2), it is always possible to chose $\Xi_1 \neq \Xi_2$ in such a way that

$$f(\mathbf{E},\Xi_1,\nabla\Xi_1) - f(\mathbf{E},\Xi_2,\nabla\Xi_2) \geqslant \partial_{\Xi} f(\mathbf{E},\Xi_2,\nabla\Xi_2)\cdot(\Xi_1-\Xi_2) + \partial_{\nabla\Xi} f(\mathbf{E},\Xi_2,\nabla\Xi_2)\cdot(\nabla\Xi_1-\nabla\Xi_2). \tag{3.4}$$

By using condition (2.24) and (3.2) again, the previous relation transforms into

$$g \geqslant c\|\Xi_1-\Xi_2\|^2 \geqslant 0, \tag{3.5}$$

where we have defined the following function:

$$g = (f(\mathbf{E},\Xi_1,\nabla\Xi_1) - f(\mathbf{E},\Xi_2,\nabla\Xi_2))\|\Xi_1-\Xi_2\|. \tag{3.6}$$

We now prove that it is possible to find a positive constant $\tilde{c}$ such that

$$\dot{g} \leqslant \tilde{c} g \tag{3.7}$$

which in turn implies that

$$g(t) \leqslant g(0)\exp(\tilde{c}t) = 0 \tag{3.8}$$

as $g(0) = 0$.[11]

[11] At the initial instant time $t = 0$, as no irreversible evolution of microcracks have been occurred, $f = 0$.

This result combined with the previous (3.5) furnishes $g = 0$ and thus, from the definition (3.6), the uniqueness condition on the microcrack density is obtained

$$\Xi_1 = \Xi_2. \tag{3.9}$$

Let us compute the time derivative of the function g:

$$\begin{aligned}\dot{g} = {} & (f(\mathbf{E}, \Xi_1, \nabla\Xi_1) - f(\mathbf{E}, \Xi_2, \nabla\Xi_2))\frac{\Xi_1 - \Xi_2}{\|\Xi_1 - \Xi_2\|}\cdot(\dot{\Xi}_1 - \dot{\Xi}_2) \\ & + (\partial_{\mathbf{E}} f(\mathbf{E}, \Xi_1, \nabla\Xi_1) - \partial_{\mathbf{E}} f(\mathbf{E}, \Xi_2, \nabla\Xi_2))\cdot\dot{\mathbf{E}}\|\Xi_1 - \Xi_2\| + \partial_{\Xi} f(\mathbf{E}, \Xi_1, \nabla\Xi_1)\cdot\dot{\Xi}_1\|\Xi_1 - \Xi_2\| \\ & - \partial_{\Xi} f(\mathbf{E}, \Xi_2, \nabla\Xi_2)\cdot\dot{\Xi}_2\|\Xi_1 - \Xi_2\| + (\partial_{\nabla\Xi} f(\mathbf{E}, \Xi_1, \nabla\Xi_1)\cdot\nabla\dot{\Xi}_1 - \partial_{\nabla\Xi} f(\mathbf{E}, \Xi_2, \nabla\Xi_2)\cdot\nabla\dot{\Xi}_2)\|\Xi_1 - \Xi_2\|.\end{aligned} \tag{3.10}$$

From condition (3.3) on the second derivatives of f, (3.1b) and (3.2), after some calculations we get, to within higher order terms in the microcrack density,

$$\begin{aligned}& (\partial_{\mathbf{E}} f(\mathbf{E}, \Xi_1, \nabla\Xi) - \partial_{\mathbf{E}} f(\mathbf{E}, \Xi_2, \nabla\Xi))\cdot\dot{\mathbf{E}} \leqslant \bar{c}\|\dot{\mathbf{E}}\|\|\Xi_1 - \Xi_2\|, \\ & (\partial_{\nabla\Xi} f(\mathbf{E}, \Xi_1, \nabla\Xi_1)\cdot\nabla\dot{\Xi}_1 - \partial_{\nabla\Xi} f(\mathbf{E}, \Xi_2, \nabla\Xi_2)\cdot\nabla\dot{\Xi}_2) \leqslant \bar{c}\|\nabla\dot{\Xi}_1 - \nabla\dot{\Xi}_2\|\|\Xi_1 - \Xi_2\|, \\ & (\partial_{\Xi} f(\mathbf{E}, \Xi_1, \nabla\Xi) - \partial_{\Xi} f(\mathbf{E}, \Xi_2, \nabla\Xi))\cdot\dot{\Xi}_1 \leqslant \bar{c}\|\dot{\Xi}_1\|\|\Xi_1 - \Xi_2\|, \\ & \partial_{\Xi} f(\mathbf{E}, \Xi_2, \nabla\Xi)\cdot(\dot{\Xi}_1 - \dot{\Xi}_2) \leqslant Cc_1\|\Xi_1 - \Xi_2\|\end{aligned} \tag{3.11}$$

which used into (3.10), together with (3.1b) and (3.6), furnish

$$\dot{g} \leqslant c_1 g + [\bar{c}(\|\dot{\mathbf{E}}\| + \|\dot{\Xi}_1\| + \|\nabla\dot{\Xi}_1 - \nabla\dot{\Xi}_2\|) + Cc_1]\|\Xi_1 - \Xi_2\|^2. \tag{3.12}$$

The last inequality by using (3.5) reduces to the following:

$$\dot{g} \leqslant \left[c_1 + \frac{Cc_1}{c} + \frac{\bar{c}}{c}(\|\dot{\mathbf{E}}\| + \|\dot{\Xi}_1\| + \|\nabla\dot{\Xi}_1 - \nabla\dot{\Xi}_2\|)\right]g \tag{3.13}$$

and finally, granted the boundedness hypothesis of velocity fields (3.1a), to (3.7).

To finally demonstrate that the solution s is unique also in term of the deformation field $\mathbf{E}$, we observe that the energy density $\hat{w}(\mathbf{E}, \Xi, \nabla\Xi)$ must be single valued, i.e. it is uniquely determined by the applied external fields $\mathscr{F}$. As a consequence, if we assume that two distinct deformation fields $\mathbf{E}_1 \neq \mathbf{E}_2$ correspond to the same data, as by (3.9) $\Xi_1 = \Xi_2 = \Xi$ and $\nabla\Xi_1 = \nabla\Xi_2 = \nabla\Xi$, the following relation must be valid:

$$\mathsf{C}[\mathbf{E}_1]\cdot\mathbf{E}_1 + 2\mathsf{H}[\Xi]\cdot\mathbf{E}_1 + 2\mathsf{f}[\nabla\Xi]\cdot\mathbf{E}_1 = \mathsf{C}[\mathbf{E}_2]\cdot\mathbf{E}_2 + 2\mathsf{H}[\Xi]\cdot\mathbf{E}_2 + 2\mathsf{f}[\nabla\Xi]\cdot\mathbf{E}_2, \tag{3.14}$$

for every choice of the constitutive tensors C, H and f compatible with the symmetry group of the material. This is possible if and only if

$$\mathbf{E}_1 = \mathbf{E}_2, \tag{3.15}$$

i.e. the solution is unique modulo an infinitesimal rigid displacement. This conclusion is achieved thanks to the presence in (3.14), behind the quadratic ones, of linear terms in the deformation. □

4. Minimum theorem

We propose a variational characterization of the linear elastic, quasi-static evolution problem of microcracked bodies based on the damage entropy flux.

In particular we consider in this Section the incremental formulation IP_1 of the problem P_1 together with an infinitesimal stability criterion of its solution. We remind that incremental stability does not imply in general neither incremental nor global uniqueness, the last one being guaranteed by hypotheses of Theorem 1.

Adopting a standard format typical of infinitesimal plasticity, for any considered function we will refer to its derivative with respect to the time parameter, denote by a superposed dot (instead of to its differential increment). It is understood that such an increment of the function is an element of the tangent space at a point of the relative manifold.

Let us start by introducing the increments of all kinematic and dynamic fields defined previously, i.e. $\dot{\mathbf{u}}, \dot{\mathbf{E}}, \dot{\mathbf{T}}, \nabla\dot{\Xi}, \dot{\mathsf{S}}, \dot{\mathbf{Z}}$, associated with the rate of the microcrack density tensor $\dot{\Xi}$, which is furnished for instance by the evolution equation (2.22). Of course by (2.5)–(2.9), (2.13) we get:

$$\dot{\mathbf{E}} = \operatorname{sym}\nabla\dot{\mathbf{u}}, \quad \nabla\dot{\Xi} = (\nabla\Xi)^{\bullet}, \tag{4.1}$$

$$\operatorname{div}\dot{\mathbf{T}} + \dot{\mathbf{b}} = 0, \quad \operatorname{div}\dot{\mathsf{S}} - \dot{\mathbf{Z}} = 0, \tag{4.2}$$

$$\mathbf{e}[\dot{\mathbf{T}}] = \mathsf{A}^{\mathrm{T}}[\dot{\mathbf{Z}}] + (\nabla\mathsf{A}^{\mathrm{T}})[\dot{\mathsf{S}}] \quad \text{in } B, \tag{4.3}$$

$$\dot{\mathbf{T}}\mathbf{n} = \dot{\mathbf{t}}, \quad \dot{\mathsf{S}}\mathbf{n} = \dot{\mathsf{t}} \qquad \text{on } \partial B, \tag{4.4}$$

together with

$$\hat{w}(\dot{\mathbf{E}}, \dot{\Xi}, \nabla\dot{\Xi}) = \frac{1}{2}(\dot{\mathbf{T}}\cdot\dot{\mathbf{E}} + \dot{\mathsf{S}}\cdot\nabla\dot{\Xi} + \dot{\mathbf{Z}}\cdot\dot{\Xi}) \tag{4.5}$$

which is the incremental energy density of the linear elastic body with microcracks.

We formulate the following variational result useful for numerical studies based on finite element schemes. We assume that all the fields are square integrable. We also assume throughout this Section the more general constitutive equations related to the damage entropy flux

$$\mathbf{h}^{\mathrm{D}} = \hat{\mathbf{h}}^{\mathrm{D}}(\mathbf{E}, \Xi, \nabla\Xi, \dot{\Xi}), \quad f = \hat{f}(\mathbf{E}, \Xi, \nabla\Xi, \dot{\Xi}) \tag{4.6}$$

in place of (2.18).

Theorem 2. *At the solution of the incremental elastic problem IP_1 for a microcracked body the functional*

$$(\mathbf{u}, \Xi, \lambda) \mapsto J(\mathbf{u}, \Xi, \lambda) \in \mathbb{R}$$

with

$$J(\mathbf{u}, \Xi, \lambda) = \int_B \hat{w}(\dot{\mathbf{E}}, \dot{\Xi}, \nabla\dot{\Xi}) - \int_B \dot{\mathbf{b}}\cdot\dot{\mathbf{u}} - \int_{\partial B}(\dot{\mathbf{t}}\cdot\dot{\mathbf{u}} + \dot{\mathsf{t}}\cdot\dot{\Xi}) + \frac{1}{2}\int_B \frac{\vartheta^2}{\rho} f^2 \dot{\lambda}^2 \tag{4.7}$$

attains a minimum.

The introduced functional is the incremental total energy with the last term accounting for the energy contribution due to nucleation, coalescence or growth of microcracks within the body. In (4.7) f is the defined configurational entropy, $\dot{\lambda}$ is the Lagrange multiplier associated with the microcrack density evolution and ρ is the mass density. This theorem is inspired by the well known Ceradini–Capurso–Maier's theorem of infinitesimal plasticity with isotropic hardening, whose generalizations to cases of general description of hardening in Cauchy continua, ideal Cosserat plasticity and strain-gradient plasticity, with the appropriate references, can be found in Mariano (2002).

Proof. To prove the statement we introduce the variations of all the incremental fields which are contained in (4.7) and assume that these variations $\delta\dot{\lambda}$, $\delta\dot{s} \equiv (\delta\dot{\mathbf{u}}, \delta\dot{\Xi}, \delta\dot{\mathbf{E}}, \delta\dot{\mathbf{T}}, \delta\nabla\dot{\Xi}, \delta\dot{\mathsf{S}}, \delta\dot{\mathbf{Z}})$ are square integrable. Of course $\nabla(\delta\dot{\Xi}) = \delta(\nabla\dot{\Xi})$ and moreover variation and rate operators commute, i.e. $\delta\dot{\phi} = (\delta\phi)^\bullet$, for every sufficiently differentiable function ϕ. We then consider the following mapping:

$$(\delta\dot{s}, \delta\dot{\lambda}) \mapsto J + \delta J \tag{4.8}$$

constructing the functional

$$J + \delta J = \int_B \hat{w}(\dot{\mathbf{E}} + \delta\dot{\mathbf{E}}, \dot{\Xi} + \delta\dot{\Xi}, \nabla\dot{\Xi} + \delta\nabla\dot{\Xi}) - \int_B \dot{\mathbf{b}} \cdot (\dot{\mathbf{u}} + \delta\dot{\mathbf{u}}) - \int_{\partial B} (\dot{\mathbf{t}} \cdot (\dot{\mathbf{u}} + \delta\dot{\mathbf{u}}) + \dot{\mathsf{t}} \cdot (\dot{\Xi} + \delta\dot{\Xi})) + \frac{1}{2}\int_B \frac{\vartheta^2}{\rho}(f + \delta f)^2(\dot{\lambda} + \delta\dot{\lambda})^2 \tag{4.9}$$

and we evaluate, by applying (4.5), the variation of J:

$$\delta J = \frac{1}{2}\int_B (\dot{\mathbf{T}} \cdot \delta\dot{\mathbf{E}} + \delta\dot{\mathbf{T}} \cdot \dot{\mathbf{E}} + \delta\dot{\mathbf{T}} \cdot \delta\dot{\mathbf{E}} + \dot{\mathsf{S}} \cdot \delta\nabla\dot{\Xi} + \delta\dot{\mathsf{S}} \cdot \nabla\dot{\Xi} + \delta\dot{\mathsf{S}} \cdot \delta\nabla\dot{\Xi} + \dot{\mathbf{Z}} \cdot \delta\dot{\Xi} + \delta\dot{\mathbf{Z}} \cdot \dot{\Xi} + \delta\dot{\mathbf{Z}} \cdot \delta\dot{\Xi}) - \int_B \dot{\mathbf{b}} \cdot \delta\dot{\mathbf{u}} - \int_{\partial B} (\dot{\mathbf{t}} \cdot \delta\dot{\mathbf{u}} + \dot{t} \cdot \delta\dot{\Xi}) + \frac{1}{2}\int_B \frac{\vartheta^2}{\rho}(\delta f)^2(\dot{\lambda} + \delta\dot{\lambda})^2 + \frac{1}{2}\int_B \frac{\vartheta^2}{\rho} f^2(\delta\dot{\lambda})^2 + \int_B \frac{\vartheta^2}{\rho} f^2 \dot{\lambda}\delta\dot{\lambda} + \int_B \frac{\vartheta^2}{\rho} f\delta f(\dot{\lambda} + \delta\dot{\lambda})^2. \tag{4.10}$$

As linear constitutive equations (2.14) are assumed to be valid, then the following relation involving macroscopic and substructural fields can be obtained:

$$\dot{\mathbf{T}} \cdot \delta\dot{\mathbf{E}} + \dot{\mathsf{S}} \cdot \delta\nabla\dot{\Xi} + \dot{\mathbf{Z}} \cdot \delta\dot{\Xi} = \delta\dot{\mathbf{T}} \cdot \dot{\mathbf{E}} + \delta\dot{\mathsf{S}} \cdot \nabla\dot{\Xi} + \delta\dot{\mathbf{Z}} \cdot \dot{\Xi} \tag{4.11}$$

by means of which (4.10) transforms into

$$\delta J = \int_B \hat{w}(\delta\dot{\mathbf{E}}, \delta\dot{\Xi}, \delta\nabla\dot{\Xi}) + \int_B (\dot{\mathbf{T}} \cdot \delta\dot{\mathbf{E}} + \dot{\mathsf{S}} \cdot \delta\nabla\dot{\Xi} + \dot{\mathbf{Z}} \cdot \delta\dot{\Xi}) - \int_B \dot{\mathbf{b}} \cdot \delta\dot{\mathbf{u}} - \int_{\partial B} (\dot{\mathbf{t}} \cdot \delta\dot{\mathbf{u}} + \dot{\mathsf{t}} \cdot \delta\dot{\Xi}) + \frac{1}{2}\int_B \frac{\vartheta^2}{\rho}(\delta f)^2(\dot{\lambda} + \delta\dot{\lambda})^2 + \frac{1}{2}\int_B \frac{\vartheta^2}{\rho} f^2(\delta\dot{\lambda})^2 + \int_B \frac{\vartheta^2}{\rho} f^2 \dot{\lambda}\delta\dot{\lambda} + \int_B \frac{\vartheta^2}{\rho} f\delta f(\dot{\lambda} + \delta\dot{\lambda})^2 \tag{4.12}$$

with

$$\hat{w}(\delta\dot{\mathbf{E}}, \delta\dot{\Xi}, \delta\nabla\dot{\Xi}) = \frac{1}{2}(\delta\dot{\mathbf{T}} \cdot \delta\dot{\mathbf{E}} + \delta\dot{\mathsf{S}} \cdot \delta\nabla\dot{\Xi} + \delta\dot{\mathbf{Z}} \cdot \delta\dot{\Xi}) \tag{4.13}$$

the incremental energy density of the linear elastic body with microcracks, evaluated at the variations.

We now suppose that there exists an equilibrated solution from which the variation $(\dot{\mathbf{u}}, \dot{\Xi}, \dot{\lambda}) \to (\dot{\mathbf{u}} + \delta\dot{\mathbf{u}}, \dot{\Xi} + \delta\dot{\Xi}, \dot{\lambda} + \delta\dot{\lambda})$ is calculated. By balance equations (4.2), (4.4) the system $(\dot{\mathbf{b}}, \dot{\mathbf{T}}, \dot{\mathbf{t}}, \dot{\mathsf{S}}, \dot{\mathbf{Z}}, \dot{\mathsf{t}})$ is *equilibrated* while $(\delta\dot{\mathbf{u}}, \delta\dot{\mathbf{E}}, \delta\dot{\Xi}, \delta\nabla\dot{\Xi})$ is a *kinematically admissible velocities* system. By applying the principle of virtual power to these systems we get the following equation:

$$\int_B \dot{\mathbf{b}} \cdot \delta\dot{\mathbf{u}} + \int_{\partial B} (\dot{\mathbf{t}} \cdot \delta\dot{\mathbf{u}} + \dot{\mathsf{t}} \cdot \delta\dot{\Xi}) = \int_B (\dot{\mathbf{T}} \cdot \delta\dot{\mathbf{E}} + \dot{\mathsf{S}} \cdot \delta\nabla\dot{\Xi} + \dot{\mathbf{Z}} \cdot \delta\dot{\Xi}) \tag{4.14}$$

by accounting for which (4.12) simplifies in

$$\delta J = \int_B \hat{w}(\delta\dot{\mathbf{E}}, \delta\dot{\Xi}, \delta\nabla\dot{\Xi}) + \frac{1}{2}\int_B \frac{\vartheta^2}{\rho}(\delta f)^2(\dot{\lambda} + \delta\dot{\lambda})^2 + \frac{1}{2}\int_B \frac{\vartheta^2}{\rho}\delta f^2(\delta\dot{\lambda})^2 + \int_B \frac{\vartheta^2}{\rho} f^2 \dot{\lambda}\delta\dot{\lambda} + \int_B \frac{\vartheta^2}{\rho} f\delta f(\dot{\lambda} + \delta\dot{\lambda})^2. \tag{4.15}$$

We observe that the first integral is not negative, as it is the integral over the body of the incremental density (4.13) which is a positive-definite form. It is then worth noting that the second and third integral in (4.15) cannot be negative.

Concerning the last two addends, finally, by remembering that $\dot{\lambda} \geqslant 0, f \geqslant 0$ and observing that during the irreversible microstructural evolution and developing of microcracks $\delta\dot{\lambda}$ and δf are non-negative functions too, i.e.

$$\delta\dot{\lambda} \geqslant 0, \quad \delta f \geqslant 0, \tag{4.16}$$

we get

$$\int_B \frac{\vartheta^2}{\rho} f^2 \dot{\lambda}\delta\dot{\lambda} \geqslant 0, \quad \int_B \frac{\vartheta^2}{\rho} f\delta f(\dot{\lambda} + \delta\dot{\lambda})^2 \geqslant 0. \tag{4.17}$$

It is then immediate to conclude that

$$J + \delta J \geqslant J, \quad \forall \delta\dot{s},\ \delta\dot{\lambda}, \tag{4.18}$$

which completes the proof. □

Remark 4. We observe that condition $(4.16)_2$ is a natural consequence of the irreversibility of a damage process. Roughly speaking, in fact, from the condition $f \geqslant 0$ on configurational entropy it results that in the variation $f + \delta f \geqslant 0$, a condition which is always fulfilled iff $\delta f \geqslant 0$, as f can also take the null value. Nevertheless condition $(4.16)_2$ can be also shown to be valid as follows.

Let us consider a regular part $\pi \subset B$, i.e. a subbody, and calculate the quantity

$$\int_\pi \dot{f}.$$

By using relations $(2.15)_1$, $(2.16)_2$ and the divergence theorem we get

$$\int_\pi \dot{f} = \left(\int_\pi \operatorname{div} \mathbf{h}^{\mathrm{D}}\right)^{\bullet} = \int_{\partial\pi} \dot{\mathbf{h}}^{\mathrm{D}} \cdot \mathbf{n} = -\int_{\partial\pi} q \geqslant 0, \quad \forall \pi \subset B. \tag{4.19}$$

By localization it results that $\dot{f} \geqslant 0$ and finally $\delta f = \dot{f}\delta t \geqslant 0$, for all $\delta t \geqslant 0$.

5. The use of damage pseudo-potentials

It is also possible to describe the evolution of microcracks within a brittle or ductile material by introducing a *damage pseudo-potential*. For the definition and existence of dissipation pseudo-potentials in damage mechanics refer to the works of Mariano and Augusti (2001) and Stumpf and Hackl (2003).

The internal self-force $\mathbf{Z}$, in fact, can be additively decomposed into two contributions,[12] the non-dissipative part (nd) and the dissipative one (d)

$$\mathbf{Z} = \mathbf{Z}^{\mathrm{nd}} + \mathbf{Z}^{\mathrm{d}} \tag{5.1}$$

[12] For viscoelastic damaged materials also the stress and the microstress dissipative parts must be taken into account, as the rate effects of the strain and of the damage variable are not negligible. Moreover the decomposition of a measure of interactions, such as the stress, the microstress and the internal self-force, into an equilibrated and a non-equilibrated part is suggested by the basic theorem of Colemam and Noll (1963) for linearly viscous materials. Also the subsequent linear dependence of the dissipative self-force on the rate of the microcrack density (cf. Eq. (5.6a)) is a rephrasing of their assumed linear dependence of the "viscous stress" on the velocity gradient.

with

$$\mathbf{Z}^{\mathrm{nd}} = \widehat{\mathbf{Z}}^{\mathrm{nd}}(\mathbf{E}, \Xi, \nabla\Xi) = \partial_{\Xi} w \tag{5.2}$$

and the dissipative part $\mathbf{Z}^{\mathrm{d}} = \widehat{\mathbf{Z}}^{\mathrm{d}}(\mathbf{E}, \Xi, \nabla\Xi, \dot{\Xi})$, which depends also on the rate of the microcrack density, such that

$$D = \widehat{\mathbf{Z}}^{\mathrm{d}}(\mathbf{E}, \Xi, \nabla\Xi) \cdot \dot{\Xi} \geqslant 0. \tag{5.3}$$

The form of this reduced dissipation inequality suggests to assume the existence of a pseudo-potential of the general form[13]

$$\gamma = \widehat{\gamma}(\mathbf{E}, \Xi, \nabla\Xi, \dot{\Xi})$$

such that the rate-dependent part of the self-force is

$$\mathbf{Z}^{\mathrm{d}} = \widehat{\mathbf{Z}}^{\mathrm{d}}(\mathbf{E}, \Xi, \nabla\Xi, \dot{\Xi}) = \partial_{\dot{\Xi}} \gamma, \tag{5.4a}$$

and

$$D = \partial_{\dot{\Xi}} \widehat{\gamma}(\mathbf{E}, \Xi, \nabla\Xi, \dot{\Xi}) \cdot \dot{\Xi} \geqslant 0. \tag{5.4b}$$

A solution of inequality (5.4b) is achieved by taking

$$\gamma = \widehat{\gamma}(\dot{\Xi}) = \frac{1}{2}\mathsf{D}[\dot{\Xi}] \cdot \dot{\Xi} \tag{5.5}$$

with D a positive-definite, fourth-order tensor, endowed with the major as well as the minor symmetries and called the *relaxation tensor*; its components are the relaxation coefficients associated with the evolution of microcracks within the body. In this case (5.4) read

$$\mathbf{Z}^{\mathrm{d}} = \mathsf{D}[\dot{\Xi}], \tag{5.6a}$$

$$D = \mathsf{D}[\dot{\Xi}] \cdot \dot{\Xi} = 2\widehat{\gamma}(\dot{\Xi}) \geqslant 0. \tag{5.6b}$$

If this constitutive linear assumption is assumed to be valid, then the evolution equation for the microcrack density can be obtained directly from the balance of substructural interactions (2.7) together with (5.1) and (5.6a):

$$\operatorname{div}\mathsf{S} - \mathbf{Z}^{\mathrm{nd}} = \mathsf{D}[\dot{\Xi}] \tag{5.7}$$

which, by taking into account $(2.14)_{2,3}$, transforms into

$$\operatorname{div}(\mathsf{S}[\nabla\Xi] + \mathsf{h}^{\mathrm{T}}[\mathbf{E}] + \mathsf{f}^{\mathrm{T}}[\Xi]) - (\mathsf{K}[\Xi] + \mathsf{H}^{\mathrm{T}}[\mathbf{E}] + \mathsf{f}[\nabla\Xi]) = \mathsf{D}[\dot{\Xi}]. \tag{5.8}$$

We name P_2 the quasi-static evolution problem we get by using a damage pseudo-potential for the dissipative part of the self-force. The obtained model is for brittle materials with viscous residuals. We immediately prove the following result.

Proposition 3. *The evolution problem* P_2 *satisfies the maximum dissipation principle provided the rate of the microcrack density* $\dot{\Xi}$ *is a constant function.*

Proof. The proof consists in the following calculation. Let us consider the constrained extremum problem: fixed $s \equiv (\mathbf{E}, \Xi, \nabla\Xi)$, maximize the dissipation function $D(\dot{\Xi})$ given by (5.6b) over the set Q, i.e.

[13] For non-isothermal processes the dependence on the temperature gradient must also be considered.

$$\sup_{\dot{\Xi}\in Q} D,$$

defined by

$$Q \equiv \{\dot{\Xi} \mid \mathbf{A} = \operatorname{div}\mathsf{S} - \mathbf{Z}^{\mathrm{nd}} - \partial_{\dot{\Xi}}\gamma = 0\}. \tag{5.9}$$

By applying the Lagrange multiplier theorem, since

$$\partial_{\dot{\Xi}}\mathbf{A} = \partial^2_{\dot{\Xi}\dot{\Xi}}\gamma = \mathbf{D} \neq 0, \tag{5.10}$$

the only possibility in order to have an extremum is that the following condition must be valid:

$$\partial_{\dot{\Xi}}\widetilde{L} = 0, \tag{5.11}$$

where we have introduced the new function

$$\widetilde{L} = \partial_{\dot{\Xi}}\gamma \cdot \dot{\Xi} + \dot{\boldsymbol{\mu}} \cdot \mathbf{A} \tag{5.12}$$

with $\dot{\boldsymbol{\mu}}$ a Lagrange multiplier, in this case a second-order constant tensor. By (5.11) and (5.12) we get

$$\partial^2_{\dot{\Xi}\dot{\Xi}}\gamma[\dot{\Xi}] + \partial_{\dot{\Xi}}\gamma - \partial^2_{\dot{\Xi}\dot{\Xi}}\gamma[\dot{\boldsymbol{\mu}}] = 0 \tag{5.13}$$

from which, by using (5.5) and the hypothesis that $\mathbf{D}$ is invertible, we have the following special value for the multiplier:

$$\dot{\boldsymbol{\mu}} = 2\dot{\Xi}. \tag{5.14}$$

Thus in order to have an extremum for D the function $\dot{\Xi}(\cdot,\cdot)$ must necessarily be constant.

After these considerations concerning the use of dissipation pseudo-potentials to describe the evolution of microcracks in brittle or quasi-brittle solids, we finally propose the following two interesting results: an uniqueness theorem and a minimum theorem for the problem P_2. Let $d_P > 0$ denotes the duration of the microcracks evolution process and $T \equiv (0, d_P]$. We remind that we consider only quasi-static processes. □

Theorem 4. *If* $\hat{w}(\mathbf{E}, \Xi, \nabla\Xi)$ *and* $\widehat{\gamma}(\dot{\Xi})$ *are positive-definite, homogeneous quadratic forms, then the evolution problem* P_2 *has a unique solution.*

Proof. Let us suppose that in the space of admissible states two distinct solutions $s_1 \equiv (\mathbf{E}_1, \Xi_1, \nabla\Xi_1) \neq s_2 \equiv (\mathbf{E}_2, \Xi_2, \nabla\Xi_2)$ of the evolution problem, satisfying the same initial conditions at time $t = 0$, correspond to the same external data $\mathscr{F}$. As equations (2.5), (2.6), (2.8), (2.14), (5.2), (5.7) are linear, by superposition $s \equiv s_1 - s_2$ is a solution of the evolution problem P_2 corresponding to null data. From the virtual work equation

$$\int_B \mathbf{b}\cdot\mathbf{u} + \int_{\partial B}(\mathbf{t}\cdot\mathbf{u} + \mathsf{t}\cdot\Xi) = \int_B (\mathbf{T}\cdot\mathbf{E} + \mathsf{S}\cdot\nabla\Xi + \mathbf{Z}^{\mathrm{nd}}\cdot\Xi + \mathbf{Z}^{\mathrm{d}}\cdot\Xi), \tag{5.15}$$

we get

$$0 = \int_B (\mathbf{T}\cdot\mathbf{E} + \mathsf{S}\cdot\nabla\Xi + \mathbf{Z}^{\mathrm{nd}}\cdot\Xi + \mathbf{Z}^{\mathrm{d}}\cdot\Xi) = \int_B 2\hat{w}(\mathbf{E}, \Xi, \nabla\Xi) + \int_0^t \int_B 2\widehat{\gamma}(\dot{\Xi}), \tag{5.16}$$

$\forall t \in (0, d_P]$, where (5.16) is obtained by using definitions (2.13), (5.5), the fact that $\|\ddot{\Xi}\| \approx 0$ and $\dot{\Xi}(\cdot, 0) = 0$. From (5.16) the conclusion that $s \equiv (0,0,0)$ and thus the solution of P_2 is unique is an immediate consequence of hypotheses contained in the thesis. □

Let $\mathscr{W}_{\mathrm{ic}}$ be the space of weak solutions which satisfy the initial conditions.

Theorem 5. *If* $\hat{w}(\mathbf{E}, \Xi, \nabla\Xi)$ *and* $\widehat{\gamma}(\dot{\Xi})$ *are positive-definite, homogeneous quadratic forms then the solution of the evolution problem* P_2 *for a microcracked body realizes the minimum of the following functional*:

$$Y : \mathscr{W}_{\mathrm{ic}} \to \mathbb{R}, \quad \textit{such that } (\mathbf{u}, \Xi) \mapsto Y(\mathbf{u}, \Xi)$$

with

$$Y(\mathbf{u}, \Xi) = \int_B [\hat{w}(\mathbf{E}, \Xi, \nabla\Xi) - \mathbf{b} \cdot \mathbf{u}] - \int_{\partial B} \mathbf{t} \cdot \mathbf{u} - \int_{\partial B} \mathrm{t} \cdot \Xi + \int_T \int_B \widehat{\gamma}(\dot{\Xi}). \tag{5.17}$$

In (5.17) the action functional

$$W_T^{\mathbf{Z}^{\mathrm{d}}}(\Xi) = -\int_T \int_B \widehat{\gamma}(\dot{\Xi}) \mathrm{d}v \, \mathrm{d}t = -\int_T \int_B \frac{1}{2} \mathbf{Z}^{\mathrm{d}} \cdot \dot{\Xi} \, \mathrm{d}v \, \mathrm{d}t \tag{5.18}$$

represents the work associated with microcracks evolution during the quasi-static damage process.

This theorem is a generalization to the present context of quasi-static evolution of anisotropic damage, where a second-order tensor is considered as damage kinematic descriptor, of the one proposed in Mosconi (2005), concerning with an isotropic model of damage quasi-static evolution, where the porosity, a scalar quantity, is the only macroscopic damage variable. The proof given there is easily adapted here, with the microvoid density replaced by the microcrack density.

6. Conclusions

The paper concerns with the multifield description of microcracks evolution in brittle or quasi-brittle solids. Two different models are considered, the first one based on the concept of damage entropy flux and the second one on that of damage pseudo-potential. In both cases uniqueness and minimum results are proved.

The proposed results should encourage the use of the multifield description of damage in brittle materials when performing numerical simulations, because they guarantee for the convergence of the related algorithm.

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